Stability of Paracontractive Open Multi-Agent Systems

D. Deplano, *Member, IEEE*, M. Franceschelli, *Senior, IEEE*, and A. Giua, *Fellow, IEEE*

***Abstract*— In this paper, we examine networks consisting of multiple interacting agents that have the flexibility to join or leave the network at any moment, which we term open multi-agent systems (OMASs). Expanding upon the recently introduced theoretical framework for analyzing the dynamic characteristics of OMASs, we extend our study to encompass agents with vector states and discrete-time evolution. A key point of our work is the employment of the concept of "open stability" w.r.t. the infinity norm, which naturally makes the distance between two points in the state independent of the number of agents. This obviates the necessity for distance normalization, as required by the standard Euclidean norm. Within this framework, the main contribution of our work is that of establishing sufficient conditions for the open stability of an OMAS, which include the boundedness of the arrival/departure process and the paracontractivity of the OMAS in the absence of arrivals/departures, thus generalizing existing results for contractive OMASs. To underscore the practical relevance of our theoretical framework, we present the formulation of the dynamic max-consensus protocol for OMASs. Through numerical simulations, we demonstrate the alignment of this protocol with the theoretical findings outlined in this manuscript.**

***Index Terms*— Autonomous Agents, Open Multi-Agent Systems, Networks, Graph Theory, Distributed Estimation.**

I. Introduction

The behavior of a large group of entities, such as robot teams, networks of computing units, sensor networks, smart grids, etc., can be captured by using the multi-agent system (MAS) paradigm. The interactions among the agents, influenced by sensing, communication, or physical coupling, are represented by a graph reflecting the network structure, where the nodes represent the agents and edges connecting the nodes represent these interactions. While most of the existing literature is limited to fixed-size networks, thus assuming that no agent may join or leave the network as time goes by, this article delves into the realm of open multi-agent systems (OMASs), where the number of agents within the network is time-varying. This characteristic is prevalent in all real engineering applications like the Internet of Things, vehicle platooning in multi-robot systems [1]–[3], energy management in smart power grids [4]–[6], online optimization in machine learning [7]–[10], consensus in cooperative networks [11]–[15], and so on.

Work supported by project e.INS- Ecosystem of Innovation for Next Generation Sardinia (cod. ECS 00000038) funded by the Italian Ministry for Research and Education (MUR) under the National Recovery and Resilience Plan (NRRP) - MISSION 4 COMPONENT 2, "From research to business" INVESTMENT 1.5, "Creation and strengthening of Ecosystems of innovation" and construction of "Territorial R& D Leaders".

D. Deplano, M. Franceschelli and A. Giua are with DIEE, University of Cagliari, 09123 Cagliari, Italy. Emails: {diego.deplano,mauro.franceschelli,giua}@unica.it

The concept of open systems was initially considered, nearly two decades ago, for networks of software agents in computer science [16], addressing problems like trust and reputation building [17], resilience against attacks [18], and law specification [19]. In contrast, it has only recently garnered attention in our control community [20]–[22], mostly due to various conceptual difficulties in adapting control-theoretic notions such as stability when the size of the systems changes over time. Indeed, simple concepts like the distance between two states become nontrivial when the two states have different dimensions. Different strategies and perspectives have been considered to overcome these issues, such as embedding the time-varying set of agents in a time-invariant superset while considering constant the state of the agents that are not currently active [23]; exploiting gossip-based interactions [24]–[26] or leveraging dwell times when agents join/leave the network [13], [27] while achieving consensus on specific metrics, such as the average, the median, and the maximum, where the influence of additive nose has been analyzed in [28]; formulating graphon models in infinite dimensional spaces to represent arbitrary-size networks of linear dynamical systems [29]–[31].

The novelty of our contribution consists in the formulation of a general stability criteria for the class of "*paracontractive*" OMASs by following the general framework presented in [32], where proper definitions of state evolution, equilibria, and stability are established for discrete-time OMASs, together with stability criteria for the special class of "*contractive*" OMASs. Differently from [32], the difficulty in evaluating the distance between vectors in different spaces $\mathcal{S}_1 \neq \mathcal{S}_2$ is overcome by using the infinity norm instead of the Euclidean norm (avoiding normalization of the distances by the number of components) and by considering only the components in the intersection of the two spaces $\mathcal{S}_1 \cap \mathcal{S}_2$ (to avoid the need to define an "*open distance*").

Structure of the paper. Section II contains background material on graph theory and generalized definitions of state evolution, equilibria, and stability for open networks. Section III provides open stability criteria for the class of "paracontractive" and "slowly expansive" systems. Section IV formulates the dynamic max-consensus problem for open networks and shows numerical simulations. Concluding remarks are given in Section V.

II. Background on Open Multi-Agent Systems

An open multi-agent system (OMAS) consists of a time-varying number n_k of interacting agents at non-negative discrete-times $k \in \mathbb{N}$. In what follows, we provide the necessary background on graph theory and open networks.

A. Open networks and graphs

The pattern of interactions among the agents in an OMAS is modeled by a time-varying undirected graph $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k)$, where $\mathcal{V}_k \subset \mathbb{N}$ is a time-varying finite set of nodes modeling the agents, and $\mathcal{E}_k \subseteq \mathcal{V}_k \times \mathcal{V}_k$ is a time-varying set of edges modeling the point-to-point communication channels between them. The number of agents $n_k = |\mathcal{V}_k| \in (0, \infty)$ is assumed strictly positive without any upper bound. Two agents i and j are said to be neighbors at time k if they share a communication channel, i.e., if there is an edge $(i,j) \in \mathcal{E}_k$. The set of neighbors of the i-th agent at time k is denoted by $\mathcal{N}_k^i = \{j \in \mathcal{V}_k : (i,j) \in \mathcal{E}_k\}$ and its cardinality gives its number of neighbors, i.e., the degree of node i at time k, denoted by $\eta_k^i = \left|\mathcal{N}_k^i\right|$. Note that graphs are assumed to be without self-loops, i.e., $i \notin \mathcal{N}_k^i$. Communications among the agents are assumed to be bidirectional, which implies that the adjacency matrix A_k is symmetric and the graph $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k)$ is undirected at any time k, i.e., for all $(i,j) \in \mathcal{E}_k$ then also $(j,i) \in \mathcal{E}_k$. The diameter δ_k of graph $\mathcal{G}_k$ at time k is the shortest distance between the two most distant nodes in the network.

Due to the time-varying nature of the network, we identify in the set of agents $\mathcal{V}_k$ the following subsets:

- *Remaining agents* $\mathcal{R}_k = \mathcal{V}_k \cap \mathcal{V}_{k-1}$: agents present in the network at time $k-1$ and time k;
- *Arriving agents* $\mathcal{A}_k = \mathcal{V}_k \setminus \mathcal{V}_{k-1}$: agents present in the network at time k but not at time $k-1$;
- *Departing agents* $\mathcal{D}_k = \mathcal{V}_k \setminus \mathcal{V}_{k+1}$: agents present in the network at time k but not at time $k+1$.

Note that, in general, the set of departing agents may contain both remaining and arriving agents, who are instead disjoint:

$$\mathcal{D}_k \subset \mathcal{V}_k = \mathcal{R}_k \cup \mathcal{A}_k, \qquad \mathcal{R}_k \cap \mathcal{A}_k = \emptyset.$$

Moreover, we assume that new arriving agents are considered as new ones and that departing agents never come back. At time k, each agent $i \in \mathcal{V}_k$ is associated with a vector state $x_k^i \in \mathbb{R}^m$ and a vector input $u_k^i \in \mathbb{R}^p$; the vectors stacking these variables are denoted by x_k and u_k, respectively. Since components of x_k and u_k are only defined for those agents in the network at time k, the sequences $\{x_k : k \in \mathbb{N}\}$ and $u_k : k \in \mathbb{N}\}$ are such that the points $x_k \in \mathbb{R}^{m|\mathcal{V}_k|}$ and $u_k \in \mathbb{R}^{p|\mathcal{V}_k|}$ have different dimensions at different times k, and thus are called "*open sequences*". The state of remaining agents in $\mathcal{R}_k$ are updated according to a causal evolution law $f^i : \mathbb{R}^{m|\mathcal{V}_k|} \times \mathbb{R}^p \to \mathbb{R}^m$, while the state of arriving agents in $\mathcal{A}_k$ need to be "initialized" according to some rule $h^i : \mathbb{R}^p \to \mathbb{R}^m$ and the state of departing agents in $\mathcal{D}_k$ are left out from x_{k+1}, yielding:

$$x_k^i = \begin{cases} f^i(x_{k-1}, u_k, \mathcal{G}_{k-1}) & \text{if } i \in \mathcal{R}_k, \\ h^i(u_k^i) & \text{if } i \in \mathcal{A}_k, \end{cases} \quad k \in \mathbb{N}\setminus\{0\}, \quad (1)$$

where x_0 and $\mathcal{G}_0$ are the initial state and the initial configuration of the network. Note that if the set of agents does not change, that is $\mathcal{V}_k = \mathcal{V}_{k-1}$, then the self-map $g_k : \mathbb{R}^{m|\mathcal{V}_k|} \to \mathbb{R}^{m|\mathcal{V}_k|}$ ruling the "*standard dynamics*":

$$x_k = g_k(x_{k-1}) := f(x_{k-1}, u_k, \mathcal{G}_{k-1}), \text{ when } \mathcal{V}_k = \mathcal{V}_{k-1}, \quad (2)$$

where $f = [f^1; \cdots ; f^{|\mathcal{V}_k|}]$.

B. Trajectory of points of interest: existence and stability

We are now ready to introduce the concept of the "*trajectory of points of interest*". This concept generalizes that of "*equilibrium point*" for time-invariant and size-invariant autonomous systems of the kind $x_k = f(x_{k-1})$, where a point $\hat{x}$ is said to be an equilibrium if $\hat{x} = f(\hat{x})$. For size-invariant systems $x_k = f(x_{k-1}, u_k)$ such that for each input u_k the system has a unique equilibrium point $\hat{x}_k = f(\hat{x}_k, u_k)$, a "*trajectory*" corresponding to a sequence of equilibrium points for different inputs u_k, $\{\hat{x}_k : k \in \mathbb{N}\}$,can be defined. For size-varying systems with inputs we can do the same with $\hat{x}_k = f(\hat{x}_k, u_k, \mathcal{G}_{k-1})$ at each $k \in \mathbb{N}$, thus resulting in a "*trajectory*" of equilibrium points of different dimensions, which we call the "*trajectory of points of interest*", originally defined by Franceschelli and Frasca in [32, Definition 3.1].

Definition 1 (Trajectory of points of interest). *Consider an OMAS and assume that the standard dynamics has a unique solution $\hat{x}_k$ at each time k, namely,*

$$\hat{x}_k = g_{k+1}(\hat{x}_k).$$

Then, the open sequence $\{\hat{x}_k : k \in \mathbb{N}\}$ is called the "trajectory of points of interest" (TPI) of the OMAS.

The existence of a TPI is guaranteed for some classes of OMASs: in this manuscript, we consider the class of paracontractive OMASs, whose trajectories exhibit a contracting distance from the TPI as time progresses.

Definition 2 (Paracontractivity). *Let $\Gamma \geq 0$, $T \geq 1$. An OMAS is said to be "(Γ, T)-paracontractive" w.r.t. $\|\cdot\|_\infty$ if there exists $\gamma \in [0,1)$ such that for all $k \geq 0$ and for all $x \in \mathbb{R}^{m|\mathcal{V}_k|}$ it holds*

$$\|(g_{k+T} \circ \cdots \circ g_{k+1})(x) - \hat{x}_k\|_\infty \leq \max\{\gamma \|x - \hat{x}_k\|_\infty, \Gamma\}, \quad (3)$$

where $\{\hat{x}_k : k \in \mathbb{N}\}$ is the TPI of the OMAS, and $\mathcal{V}_k = \cdots = \mathcal{V}_{k+T-1}$.

Remark 1. *Note that "contractive" OMASs considered in [32, Definition 3.2] – where the contraction generally refers to the distance between any two trajectories – are a special class of "paracontractive" OMASs considered in this manuscript – where the contraction refers to the distance between any trajectory and the TPI.*

Since our definition of paracontractivity allows the system to be expansive at each time step, while being paracontractive over a longer time window of length T, there is the need of having a bound on the rate of expansiveness. Thus, we also introduce the definition of "*slow expansiveness*".

Definition 3 (Slow expansiveness). *Let $\Lambda \geq 0$. An OMAS is said to be "Λ-slowly expansive" w.r.t. $\|\cdot\|_\infty$ if for* *all $k \geq 0$ and for all $x \in \mathbb{R}^{m|\mathcal{V}_k|}$ it holds*

$$\|g_{k+1}(x) - \hat{x}_k\|_\infty \leq \|x - \hat{x}_k\|_\infty + \Lambda, \tag{4}$$

where $\{\hat{x}_k : k \in \mathbb{N}\}$ is the TPI of the OMAS.

Concluding this section, we introduce a notion akin to a weak form of Lyapunov stability for OMASs. While for autonomous (time-invariant and with no inputs) and size-invariant systems the stability is a property of an equilibrium point, in our scenario of time-varying and size-varying systems the stability becomes a property of the trajectory of the point of interest, which we call "*open stability*".

Definition 4 (Open stability). *Consider an OMAS with state evolution $\{x_k : k \in \mathbb{N}\}$. Its TPI $\{\hat{x}_k : k \in \mathbb{N}\}$ is said to be "open stable" w.r.t. $\|\cdot\|_\infty$ if there is a stability radius $R \geq 0$ with the following property: for every $\varepsilon > R$, there exists $\delta > 0$ such that:*

$$\|x_0 - \hat{x}_0\|_\infty < \delta \Rightarrow \|x_k - \hat{x}_k\|_\infty < \varepsilon, \quad \forall k \geq 0.$$

Definition 5 (Global asymptotic open stability). *Consider an OMAS whose TPI $\{\hat{x}_k : k \in \mathbb{N}\}$ is open stable with stability radius $R \geq 0$. The TPI is said to be "globally asymptotically open stable" w.r.t. $\|\cdot\|_\infty$ if all trajectories converge to within a distance of R from the TPI:*

$$\limsup_{k\to\infty} \|x_k - \hat{x}_k\|_\infty \leq R.$$

We note that the use of the infinity norm $\|\cdot\|_\infty$ obviates the necessity for distance normalization by the number of agents. In contrast, when adopting any other norm $\|\cdot\|_p$ with a finite $p \geq 1$, normalization becomes imperative for ensuring a fair comparison of distances evaluated in spaces of different dimensions, as highlighted in [32, Definition 3.3] for the Euclidean norm $\|\cdot\|_2$. Indeed, when the $\|\cdot\|_\infty$ is employed, the stability radius remains bounded even if the number of agents increases over time, provided that the distance between each new agent and its corresponding component in the TPI remains bounded.

III. STABILITY OF PARACONTRACTIVE OMASS

In order to provide sufficient conditions ensuring the stability of an OMAS, in the sense of Definition 4, it is necessary to put limits on the variation of the TPI and on the process by which the agents join and leave the OMAS during time. These limits are defined next.

Definition 6 (Bounded TPI). *An OMAS with TPI $\{\hat{x}_k : k \in \mathbb{N}\}$ is said to have "bounded variation" if*

$$\exists B \geq 0: \quad \max_{r\in\mathcal{R}_k} \left\|\hat{x}^r_k - \hat{x}^r_{k-1}\right\|_\infty \leq B, \quad \forall k \in \mathbb{N}.$$

Definition 7 (Bounded arrival process). *Consider an OMAS with TPI $\{\hat{x}_k : k \in \mathbb{N}\}$. The arrival process is said to be "bounded" if*

$$\exists H \geq 0: \quad \max_{a\in\mathcal{A}_k} \|x^a_k - \hat{x}^a_k\|_\infty \leq H, \quad \forall k \in \mathbb{N}.$$

Definition 8 (OMAS dwell time). *The OMAS has dwell time $\Upsilon \in \mathbb{N}$ if changes in the number of agents are separated by at least Υ intervals of time, i.e.,*

$$\exists \Upsilon \geq 0: \quad \mathcal{V}_{k-1} \neq \mathcal{V}_k \Rightarrow \mathcal{V}_k = \cdots \mathcal{V}_{k+\Upsilon}, \quad \forall k \in \mathbb{N}.$$

We next provide a novel stability result for paracontractive OMAS, which encompasses contractive OMAS and thus extends the state-of-art [32, Theorem 3.8].

Theorem 1. *Given an OMAS, if:*

a) *it is (Γ, T)-paracontractive w.r.t. $\|\cdot\|_\infty$ and $\gamma \in (0, 1)$;*
b) *it is Λ-slowly expansive w.r.t. $\|\cdot\|_\infty$;*
c) *it admits a TPI with bounded variation with $B \geq 0$;*
d) *its arrival process is bounded with $H \geq 0$;*
e) *it has dwell time $\Upsilon \geq 0$.*

and if $\Upsilon \geq T - 1$, then the TPI is globally asymptotically open stable with radius

$$R = \rho + \min\{T - 1, 1\}(\Lambda + B).$$

where

$$\rho = \max\left\{\frac{(T-1)\Lambda + (2T-1)B}{1-\gamma}, \Gamma + TB, H\right\}.$$

Proof: Let k be a generic time and Now let $k^\star \geq k$ be the first time at which some agents join and/or leave the network, i.e., $\mathcal{V}_{k^\star-1} \neq \mathcal{V}_{k^\star}$. Thus, no agents join or leave the network before $k^\star$, i.e., $\mathcal{V}_{k-1} = \cdots = \mathcal{V}_{k^\star-1}$. By assumption b), it also holds that no agents join and/or leave the network for $\Upsilon \geq T-1$ steps from $k^\star$, i.e., $\mathcal{V}_{k^\star} = \cdots = \mathcal{V}_{k^\star+T-1}$. Let us define the time $\tau \in (k+1, k+T)$ as follows

$$\tau = \begin{cases} k^\star & \text{if } k^\star < k+T \\ k & \text{if } k^\star \geq k+T \end{cases}. \tag{5}$$

We first find an upper bound to the distance from the state trajectory and the TPI that holds at any time within $[k+1, \tau+T]$ by exploiting the slow nonexpansiveness of the OMAS. Let us compute

$$\begin{aligned}
&\|x_{k^\star} - \hat{x}_{k^\star}\|_\infty = \max_{i\in\mathcal{V}_{k^\star}} \left\|x^i_{k^\star} - \hat{x}^i_{k^\star}\right\|_\infty, \\
&= \max\{\max_{r\in\mathcal{R}_{k^\star}} \|x^r_{k^\star} - \hat{x}^r_{k^\star}\|_\infty, \max_{a\in\mathcal{A}_{k^\star}} \|x^a_{k^\star} - \hat{x}^a_{k^\star}\|_\infty\}, \\
&\overset{(i)}{\leq} \max\{\max_{r\in\mathcal{R}_{k^\star}} \|x^r_{k^\star} - \hat{x}^r_{k^\star}\|_\infty, H\}, \\
&= \max\{\max_{r\in\mathcal{R}_{k^\star}} \left\|x^r_{k^\star} - \hat{x}^r_{k^\star} \pm \hat{x}^r_{k^\star-1}\right\|_\infty, H\}, \\
&\leq \max\{\max_{r\in\mathcal{R}_{k^\star}} \{\left\|x^r_{k^\star} - \hat{x}^r_{k^\star-1}\right\|_\infty + \left\|\hat{x}^r_{k^\star} - \hat{x}^r_{k^\star-1}\right\|_\infty\}, H\}, \\
&\overset{(ii)}{\leq} \max\{\max_{r\in\mathcal{R}_{k^\star}} \left\|x^r_{k^\star} - \hat{x}^r_{k^\star-1}\right\|_\infty + B, H\},, \qquad (6) \\
&= \max\{\max_{r\in\mathcal{R}_{k^\star}} \left\|g^r_{k^\star}(x_{k^\star-1}) - \hat{x}^r_{k^\star-1}\right\|_\infty + B, H\}, \\
&\overset{(iii)}{\leq} \max\{\max_{i\in\mathcal{V}_{k^\star-1}} \left\|g^r_{k^\star}(x_k) - \hat{x}^r_{k^\star-1}\right\|_\infty + B, H\}, \\
&= \max\{\|g_{k^\star}(x_{k^\star-1}) - \hat{x}_{k^\star-1}\|_\infty + B, H\}, \\
&\overset{(iv)}{\leq} \max\{\|x_{k^\star-1} - \hat{x}_{k^\star-1}\|_\infty + \Lambda + B, H\}, \qquad (7)
\end{aligned}$$

where inequality (i) hold by assumption $a)$; inequality (ii) hold by assumption $c)$; inequality (iii) holds because $\mathcal{R}_{k+1} \subseteq \mathcal{V}_k$; inequality (iv) holds by assumption $d)$. Since by assumption the arrival/departure of agents occurs only once within $[k+1, \tau+T-1]$, from eq. (7) we conclude that for any $\Delta \in \Omega := \{1, \tau-k+T-1\}$ it holds:

$$\|x_{k+\Delta} - \hat{x}_{k+\Delta}\|_\infty \leq \max\{\|x_k - \hat{x}_k\|_\infty + (\Lambda+B), H\}. \quad (8)$$

We now proceed to find a tighter upper bound that holds only at $\tau+T-1$ by also exploiting the paracontractivity of the OMAS:

$$\begin{aligned}
&\|x_{\tau+T-1} - \hat{x}_{\tau+T-1}\|_\infty = \max_{i\in\mathcal{V}_{\tau+T-1}} \left\|x^i_{\tau+T-1} - \hat{x}^i_{\tau+T-1}\right\|_\infty, \\
&\overset{(i)}{\leq} \max\{\max_{r\in\mathcal{R}_{\tau+T-1}} \|x^r_{\tau+T-1} - \hat{x}^r_{\tau+T-2}\|_\infty + B, H\}, \\
&\overset{(ii)}{\leq} \max\{\max_{i\in\mathcal{V}_{\tau+T-2}} \left\|x^i_{\tau+T-1} - \hat{x}^i_{\tau+T-2}\right\|_\infty + B, H\}, \\
&= \max\{\max_{i\in\mathcal{V}_{\tau-1}} \left\|x^i_{\tau+T-1} - \hat{x}^i_{\tau+T-2} \pm \hat{x}^i_{\tau-1}\right\|_\infty + B, H\}, \\
&\overset{(iii)}{\leq} \max\{\max_{i\in\mathcal{V}_{\tau-1}} \left\|x^i_{\tau+T-2} - \hat{x}^i_{\tau-1}\right\|_\infty + \left\|x^i_{\tau+T-1} - \hat{x}^i_{\tau-1}\right\|_\infty + B, H\}, \\
&\overset{(iv)}{\leq} \max\{\max_{i\in\mathcal{V}_{\tau-1}} \left\|x^i_{\tau+T-1} - \hat{x}^i_{\tau-1}\right\|_\infty + TB, H\}, \\
&= \max\{\max_{i\in\mathcal{V}_{\tau-1}} \left\|(g^i_{\tau+T-1} \circ \cdots \circ g_\tau)(x_{\tau-1}) - \hat{x}^i_{\tau-1}\right\|_\infty + TB, H\}, \\
&= \max\{\|(g_{\tau+T-1} \circ \cdots \circ g_\tau)(x_{\tau-1}) - \hat{x}_{\tau-1}\|_\infty + TB, H\}, \\
&\overset{(v)}{\leq} \max\{\gamma\|x_{\tau-1} - \hat{x}_{\tau-1}\|_\infty + TB, \Gamma + TB, H\}, \\
&\overset{(vi)}{\leq} \max\{\gamma\|x_{\tau-2} - \hat{x}_{\tau-2}\|_\infty + \gamma(\Lambda+B) + TB, \Gamma + TB, H\}, \\
&\leq \ \vdots \\
&\overset{(vi)}{\leq} \max\{\gamma\|x_{k-1} - \hat{x}_{k-1}\|_\infty + \gamma(\tau-k)(\Lambda+B) + TB, \Gamma + TB, H\}, \\
&\overset{(vii)}{\leq} \max\{\gamma\|x_{k-1} - \hat{x}_{k-1}\|_\infty + \gamma(T-1)(\Lambda+B) + TB, \Gamma + TB, H\},
\end{aligned}$$

where inequality (i) holds by eq. (6) and assumptions (c)-(e); (ii) holds because $\mathcal{R}_{\tau+T-1} \subseteq \mathcal{V}_{\tau+T-2}$; (iii) holds by triangle inequality; (iv) holds by assumpion (c); (v) holds by assumption (a); (vi) holds by eq. (7) and assumption (b); (vii) holds because $\tau \in [k, k+T-1]$. The last inequality reads as:

$$\begin{aligned}
&\|x_{\tau+T-1} - \hat{x}_{\tau+T-1}\| \leq \\
&\max\{\gamma\|x_{k-1} - \hat{x}_{k-1}\|_\infty + \gamma(T-1)(\Lambda+B) + TB, \Gamma + TB, H\},
\end{aligned} \quad (9)$$

We conclude that given $\gamma < 1$ and

$$\delta = \max\left\{\frac{(T-1)\Lambda + (2T-1)B}{1-\gamma}, \Gamma + TB, H\right\},$$

then it holds

$$\|x_k - \hat{x}_k\|_\infty \leq \delta \Rightarrow \|x_{k_1} - \hat{x}_{k_1}\|_\infty \leq \delta, \quad k_1 = \tau + T. \quad (10)$$

Let us define τ_1 as in eq. (5), where k is replaced by k_1 and $k_1^\star \geq k_1$ is the new time at which some agents join and/or leave the network, then it holds

$$\|x_k - \hat{x}_k\|_\infty \leq \delta \Rightarrow \|x_{k_2} - \hat{x}_{k_2}\|_\infty \leq \delta, \quad k_2 = \tau_1 + T. \quad (11)$$

By induction, the bound holds also for all $k_1, k_2, k_3, \ldots$, defined in a similar way. Exploiting now the punctual upper bound in eq. (8) together with (9), we find an upper bound to the distance from the TPI that holds at any time after k, given by

$$R = \delta + \min\{T-1, 1\}(\Lambda + B).$$

Indeed, if $T = 1$, then $R = \delta$ because one can directly use eq. (9) to find a punctual upper bound; if instead $T > 1$, then during an interval of length T the system changes at most one, yielding an increase of the radius given by eq. (8). We conclude that the TPI is open stable with stability radius R, indeed for every $\varepsilon > R$ it holds:

$$\|x_0 - \hat{x}_0\|_\infty \leq R \Rightarrow \|x_k - \hat{x}_k\|_\infty \leq R < \varepsilon, \quad \forall k \geq 0.$$

We now prove global asymptotic open stability by considering the subsequence of states x_{k_t} for $t = 1, 2, 3, \ldots$, where k_t are defined by induction as in eqs. (10)-(11) and where the initial time is $k = 0$. Iterating eq. (9) yields

$$\begin{aligned}
\|x_{k_t} - \hat{x}_{k_t}\|_\infty \leq \max\{&\gamma^t\|x_0 - \hat{x}_0\|_\infty \\
&+ ((T-1)\Lambda + (2T-1)B)\sum_{i=0}^{t-1}\gamma^i, \Gamma + TB, H\}.
\end{aligned}$$

In the limit of $t \to \infty$, the γ^t goes to 0, and the geometric series $\sum_{i=0}^{t-1}\gamma^i$ goes to $1/(1-\gamma)$, yielding

$$\lim_{t\to\infty}\|x_{k_t} - \hat{x}_{k_t}\|_\infty \leq \max\left\{\frac{(T-1)\Lambda + (2T-1)B}{1-\gamma}, \Gamma + TB, H\right\},$$

where the term on the right-hand side is δ. Therefore, by eq. (8), we conclude that the system is globally asymptotically open stable with stability radius R, concluding the proof. ∎

Remark 2. *If $\Gamma = 0$, and $T = 1$, then the system is simply paracontractive, i.e., $\|g_{k+1}x) - \hat{x}_k\|_\infty \leq \gamma\|x - \hat{x}_k\|_\infty$. In this case, the OMAS is open stable with radius $R = \max\{B/(1-\gamma), H\}$, which is the counterpart of [32, Theorem 3.8] for the infinity norm $\|\cdot\|_\infty$.*

IV. The Dynamic Max-Consensus Problem in Open Multi-Agent Systems

Consider an OMAS in which each agent i with state $x^i_k \in \mathbb{R}^m$ and has access to a scalar time-varying reference signal $u^i_k \in \mathbb{R}$. The agents can exchange information with their neighbors according to a time-varying graph, including it can their state $y^i_k \in \mathbb{R}$ and auxiliary internal variables, however, the reference signal u^i_k is private and not exchanged.

The *dynamic max-consensus problem* consists in the design of proper local update rules for estimating and tracking the maximum $\bar{u}_k \in \mathbb{R}$ among the time-varying reference signals $\bar{u}_k = \max_{i\in\mathcal{V}_k} u^i_k$.

A. Working assumptions

Assumption 1. *The time-varying graph $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k)$ describing the pattern of interactions among agents in an OMAS is undirected and connected at all times $k \in \mathbb{N}$.*

Assumption 2. *There exists a minimum dwell time* $\Upsilon \in \mathbb{N}$ *between two consecutive changes in the set of agents, namely* $\mathcal{V}_k \neq \mathcal{V}_{k-1} \Rightarrow \mathcal{V}_k = \mathcal{V}_{k+1} = \cdots = \mathcal{V}_{k+\Upsilon}, \quad \forall k \in \mathbb{N}$.

Assumption 3. *The absolute variation of the reference signals of the agents remaining in the network is bounded by a constant* $\Pi \geq 0$, *i.e.,*

$$\forall i \in \mathcal{R}_k : \qquad |u_k^i - u_{k-1}^i| \leq \Pi, \qquad \forall k \geq 0. \tag{12}$$

In addition, we assume that join/leave events cause only bounded variations in the maximum the reference signals.

Assumption 4. *The reference signals lie within a set of size* $\Xi \geq 0$, *i.e.,*

$$|\bar{u}_k - \underline{u}_k| \leq \Xi, \qquad \forall k \geq 0. \tag{13}$$

The last assumption we make is on the time-varying diameter δ_k of the network, i.e., the longest shortest path between any two agents. Although the number $n_k = |\mathcal{V}_k|$ of the agents within the network can grow unbounded, we assume there exists an upper bound $\bar{\delta}$ on the diameter δ_k. Note that this assumption is naturally satisfied for networks with a finite number of agents, but it can hold also for networks with an infinite number of agents.

Assumption 5. *The diameter of the network is bounded by a constant* $\bar{\delta} > 0$, *i.e.,* $\delta_k \leq \bar{\delta}, \quad \forall k \geq 0$.

B. Proposed protocol and stability results

Consider a network of agents whose pattern of interaction is described by a time-varying and open undirected graph $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k)$. We denote the proposed protocol by Open Self-Tuning Dynamic Max-Consensus (OSTDMC) Protocol, which requires that the agents self-tune and exchange two additional state variables [33]: $\mu_k^i \in \mathbb{R}$ that keeps track of the maximum variation in the reference signals, and $\alpha_k^i \in \mathbb{R}$ that controls the decreasing rate of the state variable $y_k^i \in \mathbb{R}$. The state variables are stacked as follows

$$x_k^i = [y_k^i, \mu_k^i, \alpha_k^i]^\top \in \mathbb{R}^3, \qquad \forall i \in \mathcal{V}.$$

The OSTDMC Protocol is ruled by the following local updates, which makes use of two parameters $\theta \geq \beta > 0$,

$$\begin{aligned}
y_k^i &= \begin{cases} \max\limits_{j \in \mathcal{N}_{k-1}^i} \left\{ y_{k-1}^j - \operatorname*{avg}\limits_{\ell \in \mathcal{N}_{k-1}^i} \alpha_{k-1}^\ell, u_k^i \right\} & \text{if } i \in \mathcal{R}_k, \\ u_{k+1}^i & \text{if } i \in \mathcal{A}_k, \end{cases} \\
\mu_k^i &= \begin{cases} \max\limits_{j \in \mathcal{N}_{k-1}^i} \{ \mu_{k-1}^j, \theta + (u_k^i - u_{k-1}^i) \} & \text{if } i \in \mathcal{R}_k, \\ \beta & \text{if } i \in \mathcal{A}_k, \end{cases} \\
\alpha_k^i &= \begin{cases} \alpha_{k-1}^i & \text{if } i \in \mathcal{R}_k \wedge y_k^i > y_{k-1}^i, \\ \mu_k^i & \text{if } i \in \mathcal{R}_k \wedge y_k^i < y_{k-1}^i, \\ \beta & \text{otherwise.} \end{cases}
\end{aligned} \tag{14}$$

In the next theorem, we characterize the stability radius, as in Definition 4, associated to the OSTDMC. The result follows by a direct verification of all conditions of Theorem 1, thus constituting a tutorial example on how to exploit the theory on open networks proposed in this manuscript. A formal proof of this result is left to future work.

Theorem 2. *Consider an OMAS executing the OSTDMC Protocol under Assumptions 1-5 such that* $\Upsilon \geq \bar{\delta}$. *If the protocol is designed with* $\theta \geq \beta > 0$, *then the OMAS is open stable with radius* R *as in Theorem 1 where*

$$\begin{aligned}
T &= \bar{\delta} + 1, & \Gamma &= (\bar{\delta} + 1)(\theta + \Pi), \\
B &= \Pi, & \Lambda &= \theta + \Pi + \bar{\delta}\beta, \\
H &= \Xi, & \gamma &= \max\{0, \tfrac{\bar{y}_0 - \bar{u}_1 - \beta - (\Upsilon - \bar{\delta})(\theta + \Pi)}{\|y_0 - \hat{y}_0\|_\infty}\}.
\end{aligned} \tag{15}$$

C. Numerical Simulation

We now discuss a numerical simulation of the OSTDMC protocol in a network which initially has $n = 100$. At any time $k \geq 0$, a node can join the network with a probability $p_k^{join} \in [0, 1]$, establishing connections with any of the nodes in the network, or leave it with a probability $p_k^{leave} \in [0, 1]$, maintaining the network connected, such that there exists a dwell time Υ between any two of these events. In particular:

$$[p_k^{join}, p_k^{leave}] = \begin{cases} [0.4, \ 0.1] & \text{if } k \leq 2000 \\ [0.1, \ 0.8] & \text{if } k \in (2000, 3000] \\ [0.3, \ 0.3] & \text{if } k > 3000 \end{cases}.$$

The initial graph is randomly generate with diameter $\delta_0 = 5$, and we assume that such value works as an upper bound on the diameter at any time, i.e., $\delta_k \leq \bar{\delta} := 5$ for all $k \geq 0$. In turn, we consider a dwell time of $\Upsilon = 5\bar{\delta} = 25$. The initial state values of the agents y_0^i are chosen uniformly at random in the interval $[10, 11]$, while the other state variables are initialized at a very small value $\mu_0^i = \alpha_0^i = \beta := 10^{-10}$. The time-varying reference signals are sinusoidal signals given by

$$u_k^i = u_{k-1}^i + \Pi \sin\left(\frac{k}{50}\right), \qquad \Pi = 0.01,$$

whose initial value is randomly chosen in the interval $[0, 1]$. Thus, the maximum variation of the signals is $\Pi = 0.01$ and the maximum variation of the maximum signal is $\Xi = 2$. Finally, we set the minimum decreasing rate at $\theta = \Pi/2$. Therefore, Theorem 2 ensures that the OMAS is open stable with stability radius equal to $R = 2.025$. Since $\beta \approx 0$ is almost zero, it follows that the TPI is aproximately equal to a vector where all entries are equal to the maximum reference signal, $\hat{y}_{k-1} \approx \bar{u}_k \mathbf{1}$. Figures 1-2 show the evolution of the number of agents in the network, showing changes in the order of 30%, and also the distance between the state y_k of the agents and the trajectory of points of interest (TPI) $\hat{y}_k$. This distance remains bounded after decreasing during a transient phase that leads to a state of approximate consensus around the value of interest, that is the maximum reference signal. This behavior is consistent with Theorem 2, where the stability radius of $R = 2.05$ appears to be a tight estimate.

V. Conclusions

Standard system-theoretic tools do not apply directly to systems with a time-varying dimension of the state space. In this paper, we exploited the infinity norm to define the distance between states of different dimensions, then we showed that open multi-agent systems, whose dynamics (up to arrivals and departures of agents) can be defined by paracontractive maps, are stable according to our framework. This contribution generalizes previous results for contractive OMAS. Finally, we apply the results to prove the convergence properties of a novel algorithm which adapts the dynamic max-consensus protocol to open networks.

REFERENCES

[1] R. Rajamani, H.-S. Tan, B. K. Law, and W.-B. Zhang, "Demonstration of integrated longitudinal and lateral control for the operation of automated vehicles in platoons," *IEEE Transactions on Control Systems Technology*, vol. 8, no. 4, 2000.

[2] M. Franceschelli, D. Rosa, C. Seatzu, and F. Bullo, "Gossip algorithms for heterogeneous multi-vehicle routing problems," *Nonlinear Analysis: Hybrid Systems*, vol. 10, 2013.

[3] R. Patel, P. Frasca, J. W. Durham, R. Carli, and F. Bullo, "Dynamic partitioning and coverage control with asynchronous one-to-base-station communication," *IEEE transactions on control of network systems*, vol. 3, no. 1, 2015.

[4] M. Franceschelli, A. Pilloni, and A. Gasparri, "Multi-agent coordination of thermostatically controlled loads by smart power sockets for electric demand side management," *IEEE Transactions on Control Systems Technology*, vol. 29, no. 2, 2020.

[5] M. P. Fanti, A. M. Mangini, and M. Roccotelli, "A simulation and control model for building energy management," *Control Engineering Practice*, vol. 72, 2018.

[6] M. Kaheni, E. Usai, and M. Franceschelli, "Resilient and privacy-preserving multi-agent optimization and control of a network of battery energy storage systems under attack," *IEEE Transactions on Automation Science and Engineering*, 2023.

[7] K. I. Tsianos, S. Lawlor, and M. G. Rabbat, "Consensus-based distributed optimization: Practical issues and applications in large-scale machine learning," in *2012 50th annual allerton conference on communication, control, and computing (allerton)*, 2012.

[8] D. Deplano, N. Bastianello, M. Franceschelli, and K. H. Johansson, "A unified approach to solve the dynamic consensus on the average, maximum, and median values with linear convergence," in *62nd IEEE Conference on Decision and Control*, 2023.

[9] N. Bastianello, R. Carli, L. Schenato, and M. Todescato, "Asynchronous distributed optimization over lossy networks via relaxed admm: Stability and linear convergence," *IEEE Transactions on Automatic Control*, vol. 66, no. 6, 2020.

[10] N. Bastianello, D. Deplano, M. Franceschelli, and K. H. Johansson, "Robust online learning over networks," *IEEE Transactions on Automatic Control*, 2024.

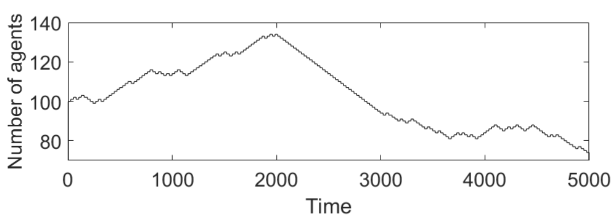

Fig. 1: Evolution of the number of agents $|\mathcal{V}_k|$.

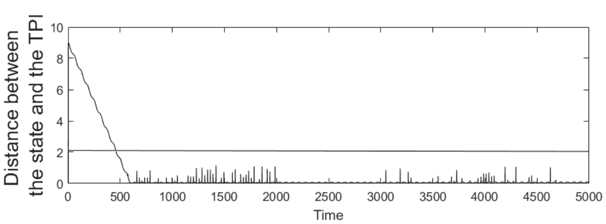

Fig. 2: Evolution of the maximum distance between the agents' state and the TPI; the horizontal line represents the stability radius $R = 2.025$.

[11] J. Török, G. Iñiguez, T. Yasseri, M. San Miguel, K. Kaski, and J. Kertész, "Opinions, conflicts, and consensus: Modeling social dynamics in a collaborative environment," *Physical review letters*, vol. 110, no. 8, 2013.

[12] D. Deplano, M. Franceschelli, and A. Giua, "Distributed tracking of graph parameters in anonymous networks with time-varying topology," in *60th IEEE Conference on Decision and Control*, 2021.

[13] D. Deplano, M. Franceschelli, and A. Giua, "Dynamic min and max consensus and size estimation of anonymous multiagent networks," *IEEE Transactions on Automatic Control*, vol. 68, no. 1, 2023.

[14] D. Deplano, M. Franceschelli, and A. Giua, "Novel stability conditions for nonlinear monotone systems and consensus in multiagent networks," *IEEE Transactions on Automatic Control*, vol. 68, no. 12, 2023.

[15] C. M. de Galland and J. M. Hendrickx, "Fundamental performance limitations for average consensus in open multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 68, no. 2, 2022.

[16] Y. Demazeau and A. R. Costa, "Populations and organizations in open multi-agent systems," in *Proceedings of the 1st National Symposium on Parallel and Distributed AI*, 1996.

[17] I. Pinyol and J. Sabater-Mir, "Computational trust and reputation models for open multi-agent systems: A review," *Artificial Intelligence Review*, vol. 40, no. 1, 2013.

[18] S. Bijani and D. Robertson, "A review of attacks and security approaches in open multi-agent systems," *Artificial Intelligence Review*, vol. 42, 2014.

[19] R. Paes, G. d. Carvalho, C. d. Lucena, P. Alencar, H. d. Almeida, and V. d. Silva, "Specifying laws in open multi-agent systems," *Agents, Norms and Institutions for Regulated Multi-agent Systems (ANIREM), AAMAS2005*, vol. 9, 2005.

[20] M. Xue, Y. Tang, W. Ren, and F. Qian, "Stability of multi-dimensional switched systems with an application to open multi-agent systems," *Automatica*, vol. 146, 2022.

[21] B. Zhou, J. H. Park, Y. Yang, Y. Jiao, and R. Hao, "Dynamic weighted average consensus of open time-varying multi-agent systems on time scales via sampled-data impulsive communication," *IEEE Transactions on Network Science and Engineering*, 2024.

[22] T. Nakamura, N. Hayashi, and M. Inuiguchi, "Cooperative learning for adversarial multi-armed bandit on open multi-agent systems," *IEEE Control Systems Letters*, vol. 7, 2023.

[23] V. S. Varma, I.-C. Morărescu, and D. Nešić, "Open multi-agent systems with discrete states and stochastic interactions," *IEEE Control Systems Letters*, vol. 2, no. 3, 2018.

[24] M. Abdelrahim, J. M. Hendrickx, and W. Heemels, "Max-consensus in open multi-agent systems with gossip interactions," in *56th IEEE Conference on Decision and Control*, 2017.

[25] J. M. Hendrickx and S. Martin, "Open multi-agent systems: Gossiping with deterministic arrivals and departures," in *2016 54th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, 2016.

[26] J. M. Hendrickx and S. Martin, "Open multi-agent systems: Gossiping with random arrivals and departures," in *2017 IEEE 56th Annual Conference on Decision and Control*, 2017.

[27] Z. A. Z. S. Dashti, C. Seatzu, and M. Franceschelli, "Dynamic consensus on the median value in open multi-agent systems," in *58th IEEE Conference on Decision and Control*, 2019.

[28] R. Vizuete, P. Frasca, and E. Panteley, "On the influence of noise in randomized consensus algorithms," *IEEE Control Systems Letters*, vol. 5, no. 3, 2020.

[29] M. Avella-Medina, F. Parise, M. T. Schaub, and S. Segarra, "Centrality measures for graphons: Accounting for uncertainty in networks," *IEEE Transactions on Network Science and Engineering*, vol. 7, no. 1, 2018.

[30] S. Gao and P. E. Caines, "Graphon control of large-scale networks of linear systems," *IEEE Transactions on Automatic Control*, vol. 65, no. 10, 2019.

[31] R. Vizuete, F. Garin, and P. Frasca, "The laplacian spectrum of large graphs sampled from graphons," *IEEE Transactions on Network Science and Engineering*, vol. 8, no. 2, 2021.

[32] M. Franceschelli and P. Frasca, "Stability of open multiagent systems and applications to dynamic consensus," *IEEE Transactions on Automatic Control*, vol. 66, no. 5, 2020.

[33] D. Deplano, M. Franceschelli, and A. Giua, "Dynamic max-consensus with local self-tuning," *IFAC-PapersOnLine*, vol. 55, no. 13, 2022.